Proof of a Riddle

Written by: Shlomi Fish

1 The Problem

We need to prove that for every natural number $n > 0$, there exists a decimal number of n digits, which can be wholly divided by 5^n, and all of its digits are odd.

2 Methodology

We will prove a stronger claim. We will demonstrate that if b_n is the corresponding number for n, then it can serve as a suffix for b_{n+1} , by adding another most significant digit.

More formally:

1. $b_1 = 5$.
2. For every n, there exists an $a \in \{1, 3, 5, 7, 9\}$ so that $b_{n+1} = b_n + a \cdot 10^n$ and $b_{n+1} \mod 5^{n+1} = 0$.

3 Proof

The proof would be by induction.

3.1 Induction Base

It holds for $n = 1$ as 5 is a one-digit number that is wholly divisable by 5^1.

3.2 Induction Step

Let's assume it holds for n and show that it also holds for $n + 1$.

Now:

$$b_{n+1} = b_n + a \cdot 10^n$$

According to the induction step b_n is wholly divisable by 5^n and so is $10^n = 5^n \cdot 2^n$. So we can divide the expression by 5^n and try to find an a so that the quotient is divisable by 5. We get:

$$b'_n + a \cdot 2^n$$

b'_n has some modulo 5, and 2^n has a non-zero modulo. The values that a can assume (1,3,5,7,9) contain all the modulos of 5. Since 5 is prime , and its modulos are a group, we can get all modulos by multiplying a given non-zero modulo by the other modulos. So we can choose an a so that the expression modulo 5 evaluates to 0. Thus we can divide this b_{n+1} by 5^{n+1} as well.

Q.E.D.